One-parameter family of additive energies and momenta in 1+1 dimensional STR

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Abstract

The velocity dependence of energy and momentum is studied. It is shown that in the case of STR in the space-time of only one spatial dimension the standard energy and momentum definition can be naturally modified without lost of local Lorenz invariance, conservation rules and additivity for multiparticle system. One parameter family of energies and momenta is constructed and it is shown that within natural conditions there is no further freedom. Choosing proper family parameter one can obtain energy and momentum increasing with velocity faster or slower in comparison with the standard case, but almost coinciding with them in the wide velocity region.

1

It is generally believed that the structure of space-time at Planck scale may differ from our everyday experience. Quantum phenomena, including quantum gravity, are expected to modify local space-time structure. Such modification can break local Lorenz invariance, can modify it or leave it untouched. A class of models exploring the last possibility is known as Double Special Relativity (DSR) [1]. Their authors, being motivated by theoretical arguments and some experimental question marks [2], introduce additional scale (connected with length or energy) and assume that the local Lorenz invariance is preserved. They pay the price of loose of standard definition of momentum and energy. In fact the price is even higher: momentum is no longer an additive quantity (momentum of a system is not a sum of momenta of its components) in DSR models; the same concerns energy [3]. The reason is clear. Simple textbook arguments [4] lead to the conclusion that the standard STR definition of momentum:

$$\mathbf{p}(m, \mathbf{v}) = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \tag{1}$$

is the unique Lorenz covariant function of $\mathbf{v}$ being additive and satisfying reasonable set of conditions. Similarly the energy function is also unique:

$$E(m, \mathbf{v}) = \frac{mc^2}{\sqrt{1 - v^2/c^2}}. \tag{2}$$

We show below that the standard arguments are not sufficient to select (1,2) in 1+1-dimensional Minkowski space. The admissible form of momentum and energy is more general and depends on a new arbitrary dimensionless parameter.

2

It is well known that the one dimensional relativistic velocity group $(V^1, \oplus)$ is commutative and associative (in contrast to the case of dimension 3). In fact the group is isomorphic with ordinary $(R, +)$ group. The group isomorphism can be written in example in the following form:

$$V^1 \ni v \to y(v) = \log \frac{1+v}{1-v} \in R \tag{3}$$

The one dimensional velocity addition is given by

$$v \oplus u = \frac{v+u}{1+vu}, \tag{4}$$

(we have assumed $c = 1$ for simplicity)

Its counterpart takes the simple linear form

$$y(v \oplus u) = y(v) + y(u). \tag{5}$$

Observe the following properties:

$$y(0) = 0, \tag{6}$$

$$y(-v) = -y(v). \tag{7}$$

These observations will be very useful for our subsequent considerations.

3

Our aim is to define additive relativistic momentum.

Assume that there exists the momentum which is a real function $p(m, v)$ that can be ascribed to each separate body (of low energy defined mass m) and that fulfils the following conditions:

1. $p(m, -v) = -p(m, v)$ (antisymmetry)
2. For a two body elastically scattering system we define the center of mass reference frame U_{CM} (it is the frame in which the asymptotic relations hold: $v_I^{in} = v_I^{CM} = -v_I^{out}$, $I = 1, 2$). We'll demand that the sum of momenta vanishes in U_{CM}:

$$p(m_1, v_1^{CM}) + p(m_2, v_2^{CM}) = 0. \tag{8}$$

3. There holds the relativistic invariant momentum conservation rule:

$$\begin{gathered} p(m_1, v_1^{CM} \oplus u) + p(m_2, v_2^{CM} \oplus u) = \\ p(m_1, -v_1^{CM} \oplus u) + p(m_2, -v_2^{CM} \oplus u) \end{gathered} \tag{9}$$

   for arbitrary u and arbitrary set of m_I, v_I^{CM} satisfying (8).

4. $p(m, v) \to mv$ for $v \ll 1$ (correspondence principle).

We will show that the most general function satisfying conditions 1.-4. is

$$p(m, v) = \frac{m}{a} \sinh\left(\frac{a}{2} \log \frac{1+v}{1-v}\right) \tag{10}$$

where a is an arbitrary real or pure imaginary parameter.

Observe that by means of the isomorphism (3) the momentum (10) can be concisely rewritten:

$$p(m, y) = \frac{m}{a} \sinh(\frac{ay}{2}). \tag{11}$$

The proof of (10) goes as follows:

Using antisymmetry of p and y one can rewrite (9)

$$\begin{aligned} & p(m_1, y_1 + w) - p(m_1, y_1 - w) + \\ & p(m_2, y_2 + w) + p(m_2, y_2 - w) = 0. \end{aligned} \tag{12}$$

where $w = y(u)$. Differentiating (12) $2n$ times with respect to the second argument and putting $w = 0$ we get the set of relations

$$p^{(2n)}(m_1, y_1) + p^{(2n)}(m_2, y_2) = 0. \tag{13}$$

Let us treat the first of them (for $n = 0$) as an involved relation between y_1 and y_2. Now, differentiating it once and twice with respect to y_1 and combining the results with (13) for $n = 1$ (in order to eliminate the second body momentum) we get

$$\frac{d}{dy_1}\left(\log \frac{dp(m_1, y_1)}{dy_1}\right) = \frac{\frac{d^2 y_2}{dy_1^2}}{\frac{dy_2}{dy_1}\left(1 - \left(\frac{dy_2}{dy_1}\right)^2\right)}. \tag{14}$$

The similar equation but with the third derivative under the logarithm, can be obtained on the similar way starting from $n = 1$. As the RHS of the both formulas are identical, we get the differential equation

$$\frac{d}{dy}\left(\log \frac{dp(m, y)}{dy}\right) = \frac{d}{dy}\left(\log \frac{d^3 p(m, y)}{dy^3}\right) \tag{15}$$

where the subscripts have been omitted.

Applying again the conditions 1. and 4. in order to fix some integration constants we get the general solution of (15) in the form given by (11).

The obtained momentum (10) comes to the standard expression (1) for $a = 1$. The plots of velocity dependencies of (10) for several values of a are compared on the Figure 1. If the value of a is close to unity, the dependencies are almost undistinguishable in the wide range of v.

The solution of (15) admits also pure imaginary parameters a. Some relevant plots are given on the Figure 2. The momenta with imaginary a are undistinguishable from the others for small velocities. However their asymptotic behavior is bizarre.

Despite the fact, that the momentum formula (10) was derived from considerations based on the elastic scattering, its applicability is universal. Consider for example a two body decay. Let M be the decaying mass and m_1, m_2 be the masses of the decay products.

Then assuming momentum conservation principle and its relativistic invariance we come to the conclusion that

$$M = m_1 \cosh(ay_1) + m_2 \cosh(ay_2). \tag{16}$$

We'll make use of this relation deriving generalized energy formula.

4

The generalized energy can be derived from the set of assumptions similar to 1.-4. Energy is expected to be a symmetric function of velocity (i), to be additive and conserved in every inertial reference frame (ii) and to have a proper low energy limit (iii). In addition it is demanded that energy is conserved in two-body decays (iv).

A similar consideration as in the case of momentum leads to the generic formula carrying out all the conditions:

$$E(m, v) = m\left((1 - \frac{1}{a^2}) + \frac{1}{a^2}\cosh\left(\frac{a}{2}\log\frac{1+v}{1-v}\right)\right) \tag{17}$$

or in an equivalent notation

$$E(m, y) = m\left((1 - \frac{1}{a^2}) + \frac{1}{a^2}\cosh\left(\frac{ay}{2}\right)\right). \tag{18}$$

It follows from the proof of (17) that the constant a is the same as previously. Again, for $a = 1$ the energy (17) takes its standard STR form (2).

The momentum and energy of a body given by (10) and (17) are connected by energy/momentum dispersion relation

$$a^2\left(E - m(1 - \frac{1}{a^2})\right)^2 - p^2 = \frac{m^2}{a^2} \tag{19}$$

which is a simple consequence of the hypergeometric unity relation.

5

The new dimensionless constant a that appears naturally in energy and momentum definitions in 1+1 dimensional space-time, allows us to modify slightly the relation between energy/momentum and velocity. For $a < 1$ both the energy and momentum are reduced in high energy limit in comparison with the standard definition. For $a > 1$ they increase. This is just the phenomenon that makes attractive DSR models. In example it helps to explain eventual ultra high energy events in cosmic radiation.

The presented analysis was done in a space-time of reduced dimension. The analysis was strictly dependent on this dimension and it is obvious that there is no straight way to generalize it. If we want to obtain similar result in the physical space-time we have to attenuate some natural conditions that usually are placed on momentum and energy. Presented analysis shows a new type of dispersion relation - in a sense alternative to the relations studied in the literature in the context of DSR.

References

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[2] K. Greisen, Phys. Rev. Lett. **16**, 748 (1966); G.T. Zatsepin and V.A. Kuzmin, JETP Lett. **4**, 78 (1966).

[3] P. Kosinski, P. Maslanka, Phys. Rev. **D68** 067702, 2003

[4] see e.g. R.P. Feynman, *Feynman lectures on Physics*, Pearson P T R (1970).

Figure Captions

Fig. 1 $p(1, v)$ plots for $a = 0.00001, 0.5, 0.7, 1, 1.2, 1.5$

Fig. 2 $p(1, v)$ plots for imaginary parameter $a = 0.5i, 0.8i, i, 1.2i, 1.5i$ compared with $a = 1$ and $a = 0.00001$. (Plot range is not symmetric.)

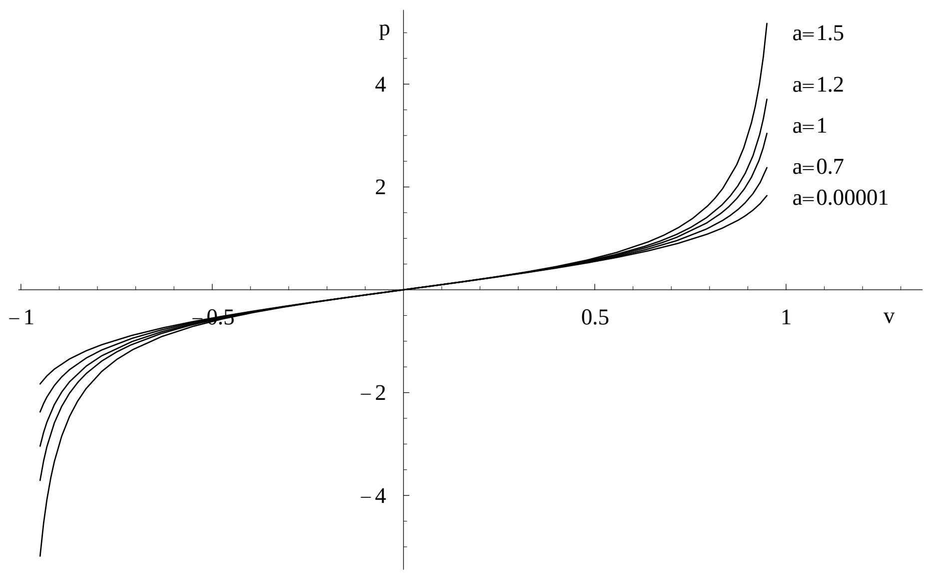

Fig. 1

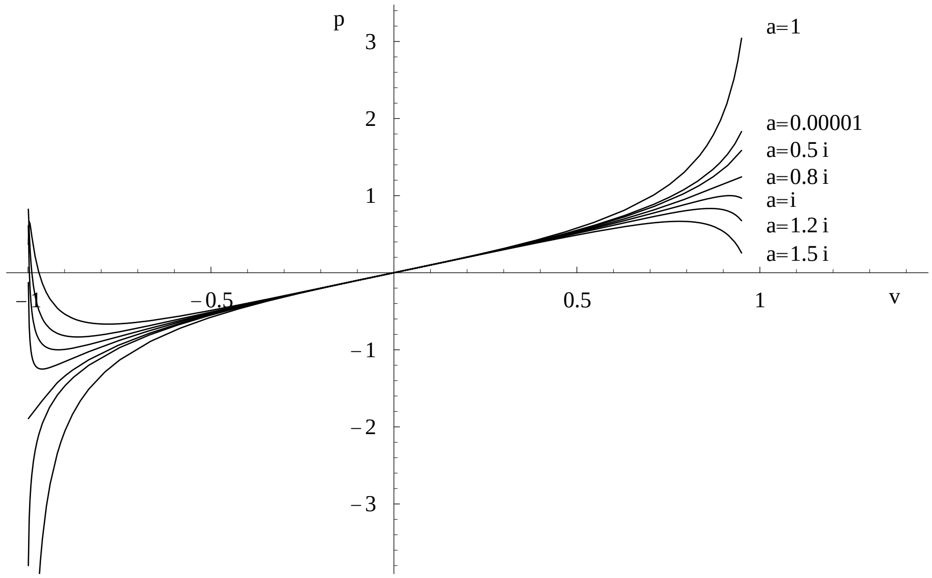

Fig. 2